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RESEARCH CONTRIBUTION

Polynomials Invertible in k-Radicals

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To the memory of Andrei Zelevinskii

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Abstract A classic result of Ritt describes polynomials invertible in radicals: they are compositions of power polynomials, Chebyshev polynomials and polynomials of degree at most 4. In this paper we prove that a polynomial invertible in radicals and solutions of equations of degree at most k is a composition of power polynomials, Chebyshev polynomials, polynomials of degree at most k and, if $k \leq 14$, certain polynomials with exceptional monodromy groups. A description of these exceptional polynomials is given. The proofs rely on classification of monodromy groups of primitive polynomials obtained by Müller based on group-theoretical results of Feit and on previous work on primitive polynomials with exceptional monodromy groups by many authors.

Keywords Topological Galois theory · Solvability in k-radicals · Exceptional polynomials

1 Introduction

This paper is devoted to a generalization of a result of Ritt on polynomials invertible in radicals:

Theorem 1 (Ritt 1922) *The inverse function of a polynomial with complex coefficients can be represented by radicals if and only if the polynomial is a composition of linear polynomials, the power polynomials $z \to z^n$, Chebyshev polynomials and polynomials of degree at most 4.*

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In the paper we give a complete description of polynomials invertible in k-radicals, i.e. in radicals and solutions of equations of degree at most k. The main result appears in Theorem 2. A more complete description of polynomials appearing in its formulation appears in Sects. 6.1–6.5.

The description of polynomials invertible in k-radicals uses deep group-theoretical result of Feit on primitive permutation groups containing a full cycle, its refinement obtained by G. Jones and work of P. Müller that builds on it to provide a classification of monodromy groups of primitive polynomials.

Description of the polynomials with primitive monodromy groups that appear in formulation of Theorem 2 is known, but scattered among many papers, and not formulated in a way that we needed for our purposes. To get a description that suited our purposes we consulted P. Müller. We are also thankful to Alexander Zvonkin who helped us understand many of the results. However we didn't obtain an exhaustive description of the exceptional polynomials of degree 15. An attempt at such a description can be found in Cassou-Noguès and Couveignes (1999) in the context of a different problem, however the results formulated in that paper are not detailed enough. We provide some information about these polynomials in 6.4 taken directly from Cassou-Noguès and Couveignes (1999).

We would like to thank P. Müller for his answers to our questions. We are especially thankful to Alexander Zvonkin, whose generous help had been of great use to us.

2 Formulation of the Problem and Its Answer

Definition 1 Let k be a natural number. A field extension L/K is k-radical if there exists a tower of extensions $K = K_0 \subset K_1 \subset \cdots \subset K_n$ such that $L \subset K_n$ and for each i, K_{i+1} is obtained from K_i by adjoining an element a_i, which is either a solution of an algebraic equation of degree at most k over K_i, or satisfies $a_i^m = b$ for some natural number m and $b \in K_i$.

Definition 2 An algebraic function $z = z(x)$ of one variable is said to be representable in k-radicals if the extension $K(z)/K$ is k-radical, where $K = F(x)$ is the field of rational functions over the base field F.

In particular an algebraic function is representable in 1-radicals if and only if it is representable in radicals.

In this paper we prove the following theorem:

Theorem 2 *A complex polynomial is invertible in k-radicals if and only if it is a composition of polynomials of degree at most k, power polynomials, Chebyshev polynomials and polynomials from the following list* (*which depends on k*):

1. *for $1 \le k \le 4$, polynomials of degree at most 4,*
2. *for $k = 5$, polynomials of degree 6 with monodromy group isomorphic to $PGL_2(5)$ with its natural action on the points of the projective line $P^1(F_5)$,*
3. *for $k = 6$, polynomials of degree 10 with monodromy group isomorphic to $P\Gamma L_2(9)$ with its natural action on the points of the projective line $P^1(F_9)$,*

4. *for $k = 7$, polynomials from list 3 above and polynomials of degree 8 with monodromy group isomorphic to $PGL_2(7)$ with its natural action on the points of the projective line $P^1(F_7)$,*
5. *for $8 \leq k \leq 14$, polynomials from list 3 and polynomials of degree 15 with monodromy group isomorphic to $PSL_4(2)$ with its natural action either on points, or hyperplanes of the projective space $P^3(F_2)$.*

Remark 1 In particular for $k \geq 15$ a polynomial is invertible in k-radicals, if and only if it is a composition of power polynomials, Chebyshev polynomials and polynomials of degree at most k.

3 Ritt's Theorem

Theorem 2 on polynomials invertible in k-radicals can be considered as a generalization of Theorem 1 of Ritt on polynomials invertible in radicals. The outline of its proof is as follows:

1. *Every polynomial is a composition of primitive ones* Every polynomial is a composition of polynomials that are not themselves compositions of polynomials of degree 2 and higher. Such polynomials are called primitive.
2. *Reduction to the case of primitive polynomials* It follows from the definition of being invertible in radicals that a composition of polynomials is invertible in radicals if and only if each polynomial in the composition is invertible in radicals. Indeed, if each of the polynomials in composition is invertible in radicals, then their composition also is. Conversely, if a polynomial R appears in the presentation of a polynomial P as a composition $P = Q \circ R \circ S$ and P^{-1} is representable in radicals, then $R^{-1} = Q \circ P^{-1} \circ S$ is also representable in radicals. Thus it is enough to classify only the primitive polynomials invertible in radicals.
3. *Galois group is responsible for representability in radicals* It follows from Galois theory that an algebraic equation over a field of characteristic zero is solvable in radicals if and only if its Galois group is solvable.
4. *A polynomial is invertible in radicals if and only if its monodromy group is solvable* A polynomial $p(x)$ is invertible in radicals if and only if the Galois group of the equation $p(x) = w$ over the field $k(w)$ is solvable. According to a result of Jordan, for $k = \mathbf{C}$ this group can be identified with the monodromy group of the function $p^{-1}(w)$.
5. *A result on solvable primitive permutation groups containing a full cycle* It follows from what we said above that a primitive polynommial is invertible in radicals if and only if its monodromy group is solvable. Since the monodromy group acts primitively on the branches of inverse of the polynomial and contains a full cycle (corresponding to a loop around the point at infinity on the Riemann sphere), the following group-theoretical result of Ritt is useful for the classification of polynomials invertible in radicals:

Theorem 3 *Let G be a primitive solvable group of permutations of a finite set X which contains a full cycle. Then either $|X| = 4$, or $|X|$ is a prime number p and X can be identified with the elements of the field F_p so that the action of G gets identified*

with the action of the subgroup of the affine group $AGL_1(p) = \{x \to ax + b | a \in (F_p)^*, b \in F_p\}$ *that contains all the shifts* $x \to x + b$.

6. *Monodromy groups of primitive polynomials invertible in radicals* It can be shown by applying Riemann–Hurwitz formula that among the groups in Theorem 3 only the following groups can be realized as monodromy groups of polynomials: (1) $G \subset S(4)$, (2) cyclic group $G = \{x \to x + b\} \subset AGL_1(p)$, (3) dihedral group $G = \{x \to \pm x + b\} \subset AGL_1(p)$.
7. *Primitive polynomials invertible in radicals* It can be easily shown (see for instance Ritt 1922; Khovanskii 2007; Burda and Khovanskii 2011) that the following result holds:

Theorem 4 *If the monodromy group of a polynomial is a subgroup of the group* $\{x \to \pm x + b\} \subset AGL_1(p)$, *then up to a linear change of variables the polynomial is either a power polynomials or a Chebyshev polynomial.*

Thus the polynomials with monodromy groups 1–3 are respectively (1) Polynomials of degree four. (2) Power polynomials up to a linear change of variables. (3) Chebyshev polynomials up to a linear change of variables.

In each of these cases the fact that the polynomial is invertible in radicals follows from solvability of its monodromy group or from explicit formulas for its inverse (see for instance Burda and Khovanskii 2011).

The outline of the proof of Theorem 2 is completely parallel to the outline discussed above. For the step 3 we use results from Khovanskii (2008), for step 5—results from Feit (1980) and Jones (2002), for step 6—results from Müller (1993), Jones (2002) and, finally, for step 7—results from Jones and Zvonkin (2002), Adrianov (1997), Cassou-Noguès and Couveignes (1999) and personal communication with P. Müller and A. Zvonkin.

4 Background on Representability in k-Radicals

It follows from the definition of a polynomial invertible in k-radicals that a composition of polynomials is invertible in k-radicals if and only if each one of the polynomials in composition is invertible in k-radicals. Thus a polynomial is invertible in k-radicals if and only if it is a composition of primitive polynomials invertible in k-radicals. In what follow we only consider primitive polynomials invertible in k-radicals.

Invertibility of a polynomial in radicals depends only on its monodromy group:

Definition 3 A group G is $[k]$-solvable if there exist subgroups $1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{n-1} \triangleleft G_n = G$ such that for each $i > 0$, G_i/G_{i-1} is either abelian, or admits a faithful action on a set with $\leq k$ elements.

It can be easily shown that this definition is equivalent to the following:

Definition 4 A group G is $[k]$-solvable if there exist subgroups $1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{n-1} \triangleleft G_n = G$ such that for each $i > 0$, G_i/G_{i-1} is a simple group, which is either abelian, or contains a subgroup of index $\leq k$.

The following result from Khovanskii (2008) describes when a field extension is k-radical:

Theorem 5 *An extension L/K of fields of characteristic zero is k-radical if and only if the Galois group $Gal(L/K)$ is $[k]$-solvable.*

In particular a polynomial is invertible in k-radicals if and only if its monodromy group is $[k]$-solvable.

5 Results of Feit, Müller and Jones

The following result on primitive permutation groups containing a full cycle had been derived by Feit as a consequence of classification of finite simple groups (Feit 1980). We provide a version of it due to Jones (2002) (the one provided in Feit 1980 was not complete in case 4):

Theorem 6 *A primitive group of permutations of n elements contains a full cycle if and only if one of the following conditions holds*:

1. $G = S_n$
2. *n is odd*, $G = A_n$ *is the group of even permutations acting naturally on n elements*,
3. *n is prime*, $C_n \subseteq G \subseteq AGL_1(n)$ *acting naturally on the field F_n, where C_n denotes a cyclic group of shifts inside the affine group $AGL_1(n)$.*
4. $n = \frac{q^d-1}{q-1}$, *where q is a power of prime and $PGL_d(q) \subseteq G \subseteq P\Gamma L_d(q)$ acting naturally either on points or on hyperplanes of the projective space $P^{d-1}(F_q)$*,
5. $n = 11$ *and* $G = PSL_2(11)$ *acting on* 11 *cosets of one of two of its subgroups of index* 11,
6. $n = 11$ *and G is Mathieu group M_{11} acting naturally on* 11 *elements*,
7. $n = 23$ *and G is Mathieu group M_{23} acting naturally on* 23 *elements.*

Using this result, and Riemann–Hurwitz formula, Müller proved the following result on monodromy groups of primitive polynomials (Müller 1993):

Theorem 7 *A group of permutations of n elements is a monodromy group of a primitive polynomial if and only if one of the following conditions holds*:

1. $G = S_n$
2. *n is odd*, $G = A_n$ *is the group of even permutations acting naturally on n elements*,
3. *n is prime*, $C_n \subseteq G \subseteq D_n = \{x \to \pm x + b \mod n\}$ *acting naturally on the field F_n*,
4. $n = 11$ *and* $G = PSL_2(11)$ *acting on* 11 *cosets of one of its subgroups of index* 11,
5. $n = \frac{p^d-1}{p-1}$, *where p is a prime number and $G = PGL_d(p)$ acting naturally either on points or on hyperplanes of the projective space $P(F_p^d)$, where (p, d) is one of the following pairs*: $(5, 2), (7, 2), (2, 3), (3, 3), (2, 4), (2, 5)$ (*in these cases n is, respectively 6,8,7,13,15,31*)
6. $n = \frac{q^d-1}{q-1}$, *where q is a power of a prime number and $G = P\Gamma L_d(q)$ acting naturally either on points or on hyperplanes of the projective space $P(F_q^d)$*,

where (q, d) *is one of the following pairs*: $(8, 2)$, $(9, 2)$, $(4, 3)$ (*in these cases* n *is, respectively 9,10,21*)
7. $n = 11$ *and* G *is Mathieu group* M_{11} *acting naturally on* 11 *elements*,
8. $n = 23$ *and* G *is Mathieu group* M_{23} *acting naturally on* 23 *elements.*

6 $[k]$-Solvable Monodromy Groups of Primitive Polynomials

According to Theorem 5 a polynomial is invertible in k-radicals if and only if its monodromy group is $[k]$-solvable, i.e. if its monodromy group G contains subgroups $1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{n-1} \triangleleft G_n = G$ such that for each $i > 0$, G_i/G_{i-1} is a simple group which is either abelian or contains a subgroup of index $\leq k$.

For each group from Theorem 7 we can determine the smallest k for which it is $[k]$-solvable:

Theorem 8 *Let* G *be a group of permutations of* n *elements, appearing in Theorem 7. The group* G *is* $[k]$*-solvable if and only if*:

1. k *is any natural number and*

   $G = S_n, n \leq 4$ *or*

   n *is prime and* $C_n \subseteq G \subseteq D_n = \{x \to \pm x + b \mod n\}$
2. $k \geq n$ *and*

   $G = S_n$, *or*

   $G = A_n$ *for odd* $n \geq 5$, *or*

   $G = PSL_2(11)$ *or* $G = M_{11}$ *for* $n = 11$, *or*

   $G = M_{23}$ *for* $n = 23$, *or*

   $G = PGL_3(2)$ *for* $n = 7$, *or*

   $G = PGL_3(3)$ *for* $n = 13$, *or*

   $G = PGL_5(2)$ *for* $n = 31$, *or*

   $G = P\Gamma L_3(4)$ *for* $n = 21$, *or*

   $G = P\Gamma L_2(8)$ *for* $n = 9$,

3. $G = PGL_2(5), k \geq 5$,
4. $G = P\Gamma L_2(9), k \geq 6$,
5. $G = PGL_2(7), k \geq 7$,
6. $G = PGL_4(2), k \geq 8$.

Proof Let G be a finite group and let $\{e\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$ be its composition series. Then the smallest k for which G is $[k]$-solvable is the smallest k for which all the composition factors G_{i+1}/G_i are either abelian or contain a proper subgroup of index at most k.

The group A_n, $n \geq 3$ doesn't contain a proper subgroup of index smaller than n (otherwise A_n can be embedded in S_k for $k < n$ and $n!/2$ is $< k!$).

The group A_n is a composition factor of groups S_n and A_n, $n \geq 5$ from Theorem 7, and hence these groups are $[k]$-solvable only for $k \geq n$.

The simple groups M_{11} and M_{23} don't have a proper subgroup of index smaller than 11 and 23 respectively (Conway et al. 1985), and thus they are k-solvable only for $k \geq 11$ and $k \geq 23$ respectively.

Compositional factors of groups $PGL_n(q)$ and $P\Gamma L_n(q)$ (for $n \geq 2$ and $q \neq 2, 3$) are either abelian or isomorphic to the simple group $PSL_n(q)$, as can be seen from the natural homomorphisms onto abelian groups $P\Gamma L_n(q) \to \mathrm{Aut}(F_q)$ with kernel $PGL_n(q)$ and $PGL_n(q) \xrightarrow{\det} F_q^*/(F_q^*)^n$ with kernel $PSL_n(q)$ ($(F_q^*)^n$ is the subgroup of invertible elements of F_q that are n-th powers). For small n and q the smallest index of a proper subgroup of $PSL_n(q)$ can be found in Conway et al. (1985) (we use the notation $L_n(q)$ for $PSL_n(q)$).

G	$L_2(5)$	$L_2(7)$	$L_3(2)$	$L_2(11)$	$L_2(8)$	$L_2(9)$	$L_3(3)$	$L_4(2)$	$L_3(4)$	$L_5(2)$
k	5	7	7	11	9	6	13	8	21	31

In all the cases except $L_2(5)$,$L_2(7)$,$L_2(9)$,$L_4(2)$ this k coincides with the number of elements on which the corresponding group from Theorem 7 acts.

In cases $L_2(5)$,$L_2(7)$,$L_2(9)$,$L_4(2)$ one has the following exceptional isomorphisms: $PSL_2(F_5) = A_5$, $PSL_2(F_7) = PSL_3(F_2)$, $PSL_2(F_9) = A_6$, $PSL_4(F_2) = A_8$. □

A polynomial of prime degree with cyclic or dihedral monodromy group is, up to a linear change of variables, a power polynomial or Chebyshev polynomial respectively. Thus we obtain the following theorem:

Theorem 9 *A primitive polynomial is invertible in k-radicals if and only if it has degree at most k, or one of the following conditions holds*:

1. $1 \leq k$, *the degree of the polynomial is a prime number and up to a linear change of variables the polynomial is a power polynomial or Chebyshev polynomial*,
2. $k \leq 3$, *the degree of the polynomial is 4*,
3. $k = 5$, *the degree of the polynomial is 6 and its monodromy group is* $PGL_2(5)$,
4. $6 \leq k \leq 9$, *the degree of the polynomial is 10 and its monodromy group is* $P\Gamma L_2(9)$,
5. $k = 7$, *the degree of the polynomial is 8 and its monodromy group is* $PGL_2(7)$,
6. $8 \leq k \leq 14$, *the degree of the polynomial is 15 and its monodromy group is* $PGL_4(2)$.

The polynomials appearing in the exceptional cases 3–6 can be described explicitly: in cases 3–5 there is only a finite number of such polynomials up to a linear change of

variables, while in case 6 equivalence classes of such polynomials up to linear change of variables form two one-parametric families. Below we describe such polynomials. To describe these polynomials we will use the following notions (see Lando and Zvonkin 2013; Khovanskii and Zdravkovska 1996):

Definition 5 The *passport* of a polynomial of degree n is the set of partitions of n corresponding to the cycle decompositions of the local monodromy operators at the polynomial's finite ramification points.

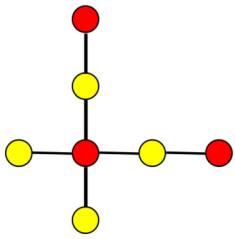
Definition 6 The *Dessin d'enfant* of a polynomial having two finite ramification points is a planar graph whose vertices are the preimages of the finite ramification points of the polynomial, and whose edges correspond to preimages of an interval connecting the two finite ramification points.

6.1 Polynomials, Invertible in 5-Radicals

According to Theorem 9, polynomials invertible in 5-radicals are compositions of power polynomials, Chebyshev polynomials, polynomials of degree at most 5 and polynomials of degree 6 with monodromy group isomorphic to the group $PGL_2(5)$ with its natural action on the 6 points of the projective line over the field F_5 (the dual action of the group $PGL_2(5)$ on hyperplanes of the projective line over F_5 is the same as the action on points, since in this case hyperplanes are in fact just points).

Theorem 10 *A primitive polynomial of degree six is invertible in 5-radicals if and only if one of the following conditions holds*:

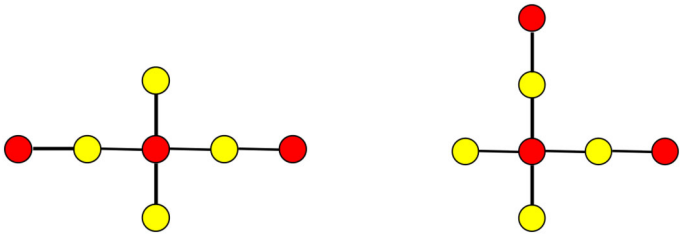
- *The monodromy group of the polynomial is isomorphic to the group* $PGL_2(5)$ *with its natural action on* $P^1(F_5)$
- *The passport of the polynomial is* $[2^2 1^2, 4^1 1^2]$
- *Dessin d'enfant of the polynomial is*



- *By means of an affine change of variables the polynomial can be brought to the form* $p(z) = z^4(z^2 + 6z + 25)$

Proof The permutations of 6 elements given by the action of $PGL_2(5)$ on $P^1(F_5)$ have cycle structures 1^6, $2^2 1^2$, 2^3, $4^1 1^2$, 3^2, $5^1 1^1$, 6^1. Since the derivative of a polynomial of degree 6 has 5 roots counted with multiplicities, the cycle structures of permutations corresponding to small loops around the critical values must be either $2^2 1^2$, 2^3, or $2^2 1^2$, $4^1 1^2$. The first choice corresponds (according to Theorem 18 from Khovanskii

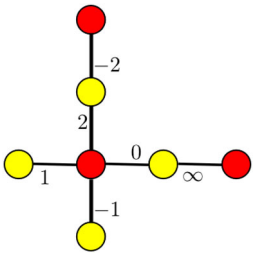
2007) to the case of Chebyshev polynomial. A polynomial with passport $[2^2 1^2, 4^1 1^2]$ can have one of the two dessins d'enfant:



A polynomial with the first dessin d'enfant is a composition of a polynomial of degree 3 and a polynomial of degree 2.

The monodromy group of a polynomial with the second dessin d'enfant is $PGL_2(5)$.

Indeed, if one labels the edges of the dessin as in the picture below,

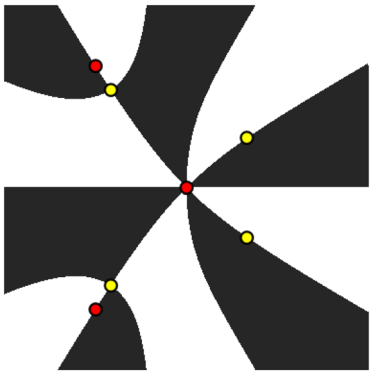


then the small loops around the critical values going in counterclockwise direction correspond to the permutations $x \to \frac{1}{x} \mod 5$ and $x \to 2x + 2 \mod 5$, which generate the group $PGL_2(5)$.

We now show that by an affine change of variables a polynomial of degree six with monodromy group $PGL_2(5)$ can be brought to the form $z^4(z^2 + 6z + 25)$.

As we have found above, the passport of such a polynomial p is $[2^2 1^2, 4^1 1^2]$. By an affine change of coordinates one can make the point of multiplicity 4 to be at zero and make the polynomial vanish at this point. One can also make the leading coefficient of the polynomial be 1. Then the polynomial has the form $p(z) = z^4(z^2 + az + b)$. Its derivative is $p'(z) = z^3(6z^2 + 5az + 4b)$. The values of the polynomial p at the zeroes of the factor $6z^2 + 5az + 4b$ must be equal, and hence the remainder of division of p by $6z^2 + 5az + 4b$ must be a constant polynomial. The coefficient at z of the remainder of division of p by $6z^2 + 5az + 4b$ is $\frac{1}{6^5}a(96b - 25a^2)(36b - 25a^2)$. If $a = 0$, then the polynomial p is a composition of a polynomial of degree 3 and the polynomial z^2. If $96b = 25a^2$, then $6z^2 + 5az + 4b$ is a perfect square, and hence the passport of p is not $[2^2 1^2, 4^1 1^2]$. Finally if $36b = 25a^2$, then by a linear change of variables one can make p to be the polynomial $z^4(z^2 + 6z + 25)$ with critical values 0 and $-\frac{2^4 5^5}{3^3}$.

A picture of the dessin d'enfant of this polynomial on which the preimage of the upper half-plane is colored black (and red and yellow dots are the preimages of the critical values) is as follows:



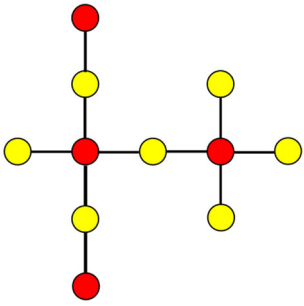
□

6.2 Polynomials Invertible in 6-Radicals

According to Theorem 9, polynomials invertible in 6-radicals are compositions of power polynomials, Chebyshev polynomials, polynomials of degree at most 6 and polynomials of degree 10 with monodromy group isomorphic to the group $P\Gamma L_2(9)$ with its natural action on the 10 points of projective line over the field with nine elements F_9.

Theorem 11 *A primitive polynomial of degree 10 is invertible in 6-radicals if and only if one of the following conditions holds*:

- *The monodromy group of the polynomial is isomorphic to the group* $P\Gamma L_2(9)$ *with its natural action on* $P^1(F_9)$
- *The dessin d'enfant of the polynomial is*



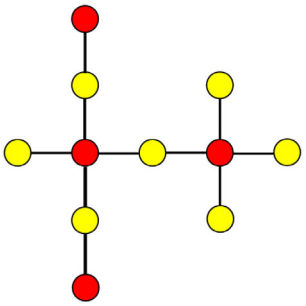
- *By means of an affine change of variables the polynomial can be brought to the form* $p(z) = \left(z^2 - \frac{81}{500}\right)^4 \left(z^2 + z + \frac{189}{500}\right)$

Proof One can check Müller (1993), p. 10, that the only possible passport of a polynomial of degree 10 with monodromy group $P\Gamma L_2(9)$ is the passport $[2^3 1^4, 4^2 1^2]$.

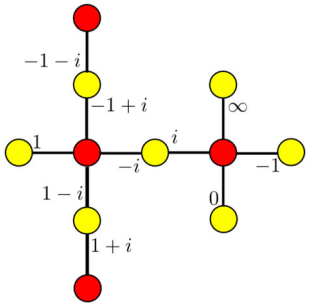
We will let i denote an element $i \in F_9$ satisfying $i^2 = -1$. We will also denote the Frobenius automorphism of the field F_9 by $x \to \overline{x}$.

The group $P\Gamma L_2(9)$ acting on 10 elements of the projective line over the field F_9 contains only one conjugacy class of a 10-cycle: it is the class C_1 of the element $\frac{1+x}{i-x}$. It also contains only one conjugacy class C_2 of an element with cycle structure $2^3 1^4$: it is the class of the element $x \to \overline{x}$. There are two conjugacy classes of elements with cycle structure $1^2 4^2$: the class C_3 of element $x \to (1+i)\overline{x}$ and the class $\overline{C}_3$ of the element $x \to (1+i)x$. Only the class C_3 can correspond to local monodromy of our polynomial, since the product of elements of classes $C_1, \overline{C}_3$ belongs to the subgroup $PGL_2(9)$ of the group $P\Gamma L_2(9)$, and thus can't belong to C_2. One can verify that there exists only one solution (up to conjugacy) of the equation $\sigma_1\sigma_2\sigma_3 = 1$ with $\sigma_i \in C_i$: $\sigma_1 = x \to \frac{1+x}{i-x}$,$\sigma_2 = x \to \overline{x}$,$\sigma_3 = x \to \frac{i\overline{x}-1}{\overline{x}+1}$. Thus the branching data for our polynomial are rigid (Völklein 1996), Definition 2.15. Hence our polynomial is defined over the rationals (Völklein 1996), Theorem 3.8.

It follows from the above considerations that the dessin d'enfant of the polynomial of degree 10 with monodromy group $P\Gamma L_2(9)$ is as follows:



Conversely, the monodromy group of a polynomial with such dessin is isomorphic to $P\Gamma L_2(9)$, because one can label the edges of the dessin with elements of $P^1(F_9)$ as follows:



Then local monodromies around the critical values correspond to the permutations $x \to \overline{x}$ (corresponding to yellow vertices in the figure) and $x \to \frac{i\overline{x}-1}{\overline{x}+1}$ (corresponding to red vertices in the figure), which generate the group $P\Gamma L_2(9)$.

We now show that using an affine change of variables the polynomial can be brought to the form $\left(z^2 - \frac{81}{500}\right)^4 \left(z^2 + z + \frac{189}{500}\right)$. By means of a change of variables defined over the rationals one can make sure that the critical value corresponding to the critical points of order 4 is zero. One can also make the average of the these two critical points be at zero. By means of a further change of variables one can bring the polynomial to the form $p(z) = (z^2 - a)^3(z^2 + z + b)$. In this case $p'(z) = (z^2 - a)^3(10z^3 + 9z^2 + (8b - 2a)z - a)$. Since the values of p at the zeroes of the polynomial $q_3(z) = 10z^3 + 9z^2 + (8b - 2a)z - a$ must be equal, the remainder from division of p by q_3 must be a constant polynomial. Equating the coefficients at z and z^2 of this remainder to zero and eliminating the variable b we find that the value of a can be equal either to $\frac{-27}{100}$, or to $\frac{81}{500}$, or to a root of a polynomial of degree 5 or 9, that is irreducible over the rationals.

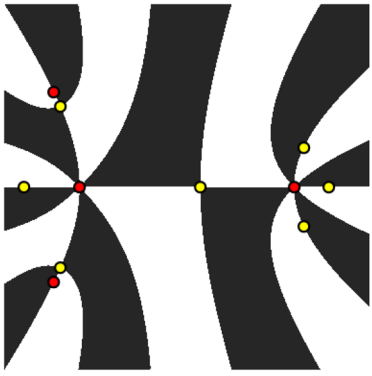
The value $a = -\frac{27}{100}$ corresponds to the case when q_3 is a perfect cube, in which the passport of the polynomial p is not the one that we want.

The value $a = \frac{81}{500}$ corresponds to $b = \frac{189}{500}$.

The cases when a is a root of irreducible over $\mathbf{Q}$ polynomials of degree 5 or 9 correspond to polynomials with monodromy groups different from $P\Gamma L_2(9)$ (we have seen above that our polynomial is defined over $\mathbf{Q}$).

Thus by means of an affine change of variables the polynomial can be made equal to the polynomial $\left(z^2 - \frac{81}{500}\right)^4 \left(z^2 + z + \frac{189}{500}\right)$ with critical values 0 and $\frac{2^4 3^{12}}{5^{15}}$.

A picture of the dessin d'enfant of this polynomial on which the preimage of the upper half-plane is colored black (and red and yellow dots are the preimages of the critical values) is as follows:



□

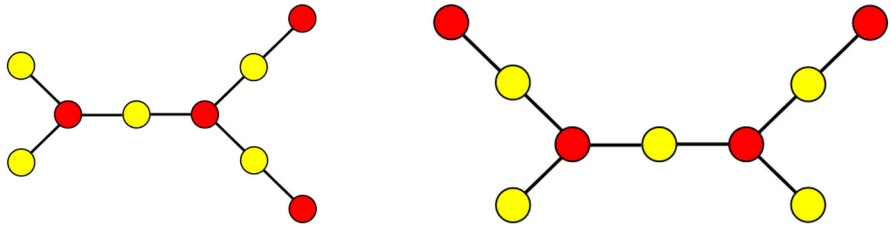
6.3 Polynomials Invertible in 7-Radicals

According to Theorem 9, polynomials invertible in 7-radicals are compositions of power polynomials, Chebyshev polynomials, polynomials of degree at most 7, poly-

nomials of degree 10 with monodromy group isomorphic to $P\Gamma L_2(9)$ described in Sect. 6.2, and polynomials of degree 8 with monodromy group isomorphic to $PGL_2(7)$ with its natural action on the 8 points of the projective line over the field F_7.

Theorem 12 *A primitive polynomial of degree 8 is invertible in 7-radicals if and only if one of the following conditions holds*:

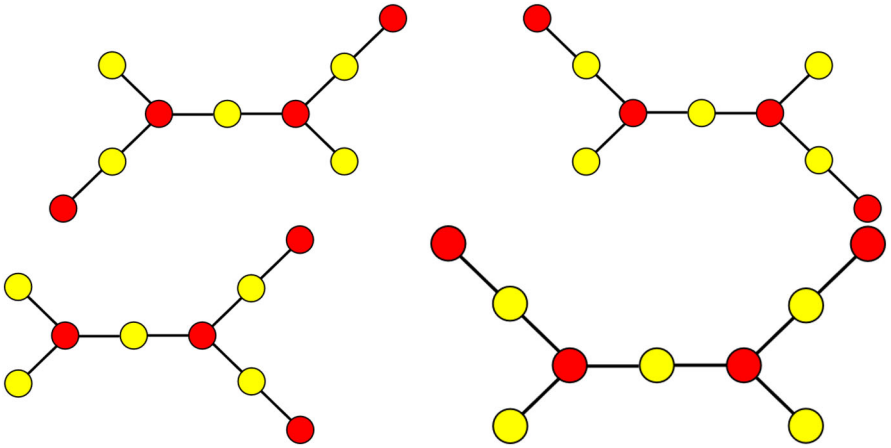
- *The monodromy group of the polynomials is isomorphic to the group $PGL_2(7)$ with its natural action on $P^1(F_7)$,*
- *The dessin d'enfant of the polynomial is one of the following*:



- *By means of an affine change of variables the polynomial can be brought to the form $p(z) = (z^2 + \frac{25+22\sqrt{2}}{64})^3(z^2 + z + \frac{97+54\sqrt{2}}{64})$ or to the form $p(z) = (z^2 + \frac{25-22\sqrt{2}}{64})^3(z^2 + z + \frac{97-54\sqrt{2}}{64})$.*

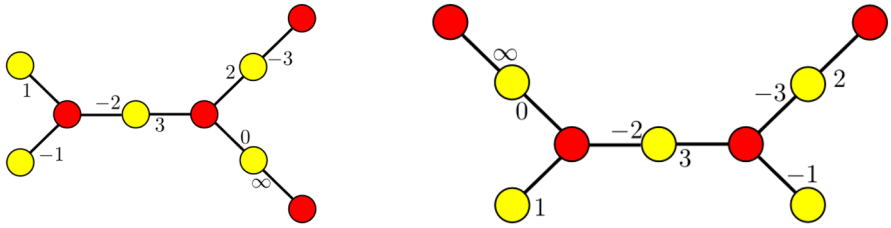
Proof One can verify Müller (1993), p. 6, that the only possible passport of a polynomial of degree 8 with monodromy group $PGL_2(7)$ is the passport $[2^3 1^2, 3^2 1^2]$.

A polynomial with this passport can have one of the following dessins:



A polynomial with one of the top two dessins is a composition of polynomials of degree 2 and degree 4 (indeed, each dessin is invariant under rotation by 180°).

The monodromy groups of polynomials with the bottom two dessins are isomorphic to the group $PGL_2(7)$. Indeed, if the edges of the dessins are labelled by elements of $P^1(F_7)$ as on the pictures below,



then the local monodromies around the critical values correspond to the permutations $x \to \frac{1}{x}$ and $x \to 2 - 3x$ for the first of them, and to the permutations $x \to \frac{1}{x}$ and $x \to 1 - 3x$ for the second. In each case they generate the group $PGL_2(7)$.

The group $PGL_2(7)$, acting on 8 elements of the projective line over F_7 contains two conjugacy classes of 8-cycles: the class C_1 of the element $\frac{3}{2-x}$ and the class C_1' of the element $\frac{3}{1-x}$. It contains one conjugacy class C_2 of an element with cycle structure $1^2 2^3$: it is the class of the element $x \to \frac{1}{x}$. There is one conjugacy class of an element with cycle structure $1^2 3^2$: the class C_3 of the element $x \to 2x$. One can show that up to conjugacy there is only one solution of the equation $\sigma_1\sigma_2\sigma_3 = 1$ with $\sigma_i \in C_i$ (namely $\sigma_1 = x \to \frac{3}{2-x}$, $\sigma_2 = x \to \frac{1}{x}$, $\sigma_3 = x \to 2 - 3x$). Also there is only one solution of the equation $\sigma_1\sigma_2\sigma_3 = 1$ with $\sigma_1 \in C_1'$,$\sigma_2 \in C_2$,$\sigma_3 \in C_3$ (namely $\sigma_1 = x \to \frac{3}{1-x}$, $\sigma_2 = x \to \frac{1}{x}$, $\sigma_3 = x \to 1 - 3x$).

Thus the branching data for our polynomial are rigid. The 8-cycle is defined over an extension of $\mathbf{Q}$ by a root of unity of order 8. Thus our polynomial is defined over the extension $\mathbf{Q}(\sqrt{2}, i)$ of degree 4 over $\mathbf{Q}$.

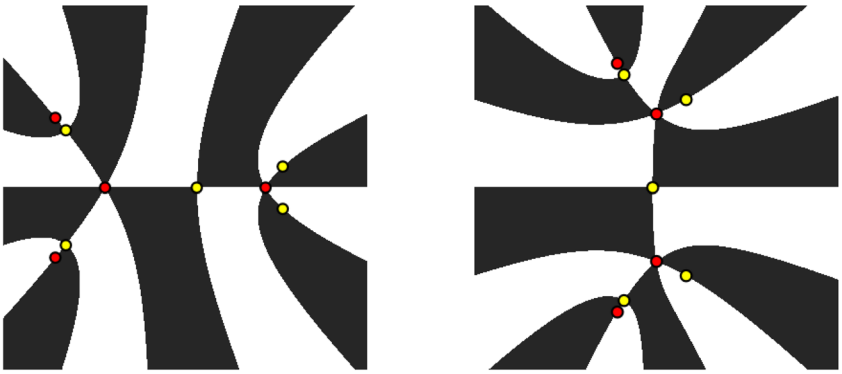
As in the previous section, we can assume that the polynomial has the form $p(z) = (z^2 - a)^3(z^2 + z + b)$. Then $p'(z) = (z^2 - a)^2(8z^3 + 7z^2 + (6b - 2a)z - a)$. Since the values of the polynomial p at the zeroes of the polynomial $q_3(z) = 8z^3 + 7z^2 + (6b - 2a)z - a$ must be equal, the remainder from division of p by q_3 must be a constant polynomial. Equating the coefficients at z and z^2 of this remainder to zero and eliminating the variable b we find that either $a = -\frac{343}{1728}$, or $4096a^2 + 3200a - 343 = 0$, or a is a root of a polynomial of degree 6 that is irreducible over the rationals.

The value $a = -\frac{343}{1728}$ corresponds to the case when q_3 is a perfect cube. In this case the passport of the polynomial is not the one we are looking for.

The value $a = \frac{-25 \pm 22\sqrt{2}}{64}$ corresponds to $b = \frac{97 \mp 54\sqrt{2}}{64}$.

The case when a is a root of an irreducible degree 6 polynomial over $\mathbf{Q}$ corresponds to polynomial with monodromy group different from $PGL_2(7)$ (our polynomial is defined over $\mathbf{Q}(\sqrt{2}, i)$).

Pictures of the dessin d'enfants of these polynomials on which the preimage of the upper half-plane is colored black (and red and yellow dots are the preimages of the critical values) is as follows:



□

6.4 Polynomials Invertible in k-Radicals, $8 \leq k \leq 14$

According to Theorem 9, polynomials invertible in k-radicals for $8 \leq k \leq 14$ are compositions of power polynomials, Chebyshev polynomials, polynomials of degree at most k, polynomials of degree 10 with monodromy group isomorphic to $P\Gamma L_2(9)$ described in Sect. 6.2 and polynomials of degree 15 with monodromy group isomorphic to $PGL_4(2)$ with its natural action either on the 15 points or on the 15 hyperplanes of the thee-dimensional projective space over the field F_2.

Polynomials of degree 15 with monodromy group isomorphic to $PGL_4(2)$ can have one of the following passports (Jones and Zvonkin 2002; Adrianov 1997): $[2^6 1^3, 2^4 1^7, 2^4 1^7]$, $[4^3 2^1 1^1, 2^4 1^7]$, $[4^2 2^2 1^3, 2^6 1^3]$, $[6^1 3^2 2^1 1^1, 2^4 1^7]$.

Such polynomials had been investigated in Cassou-Noguès and Couveignes (1999) in context of finding pairs of polynomials g, h such that the curve $g(x) = h(y)$ is reducible. In Cassou-Noguès and Couveignes (1999) it is proved that a polynomial with monodromy group isomorphic to $PGL_4(2)$ and passport $[2^6 1^3, 2^4 1^7, 2^4 1^7]$ can be brought by an affine change of variables to the form

$$
\begin{aligned}
g_t^a(x) = {} & \frac{x^{15}}{15} + (a-1)tx^{13} + (a+7)tx^{12} - (5a+21)t^2x^{11} + 2(37a-71)t^2x^{10} \\
& - \frac{(261a-349)(151598t+141075a-109260)t^2}{454794}x^9 \\
& - (649a+703)t^3x^8 \\
& + \frac{3(46a+239)(76579t+198260a-462560)t^3}{76579}x^7 \\
& - \frac{4(548a-1939)(259891t+106365a-26420)t^3}{259891}x^6
\end{aligned}
$$

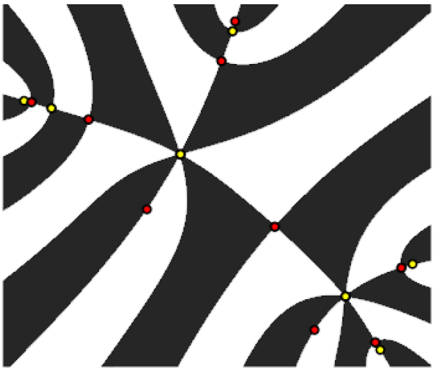
$$+\frac{3(1945a-1581)(7278308t+14685825a-113700500)t^4}{36391540}x^5$$
$$+\frac{3(3233a+2051)(877444t+1339725a-2162500)t^4}{877444}x^4$$
$$+\frac{9(9a-133)\left(50448t^2-162040at-320375a-1260960t+23500\right)t^4}{16816}x^3$$
$$+\frac{9(403a-1559)(5108t+9165a-39620)t^5}{2554}x^2$$
$$-\frac{135}{16}(7a+5)(4t-75a-100)(4t+5a-4)t^5x$$
$$+675(a-8)(t-16)t^6,$$

where a is one of the two roots of the equation $a^2-a+4=0$ and t is a complex number.

This result is mentioned there only briefly and leaves several questions unanswered:

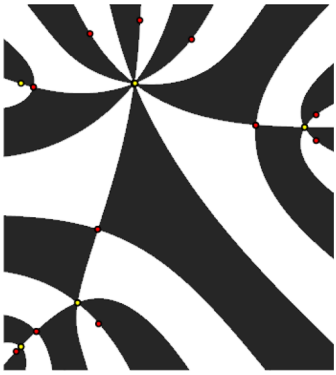
Do all the polynomials from the families g_t^a have monodromy group $PGL_4(2)$ with action on the points or on the hyperplanes of the space $P^3(F_2)$ depending on the choice of a for all parameters $t\neq 0$?

Can all the polynomials of degree 15 with monodromy group isomorphic to $PGL_4(2)$ can be brought by an affine change of variables to the form $g_t^a(x)$ for some t and some choice of a? In particular do all the polynomials with monodromy group $PGL_4(2)$ with passports $[4^3 2^1 1^1, 2^4 1^7]$, $[4^2 2^2 1^3, 2^6 1^3]$, $[6^1 3^2 2^1 1^1, 2^4 1^7]$ correspond to some values of the parameter? It is certainly true for some of them. For instance for $t=75/4$ the polynomial g_t^a has passport $[4^2 2^2 1^3, 2^6 1^3]$ and dessin d'enfant



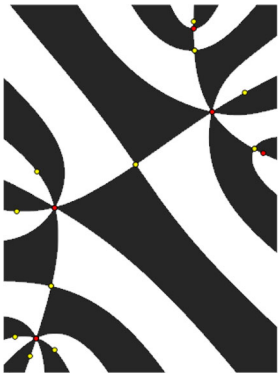
(or its reflection for the other choice of a).

For $t=-5/4$ the polynomial has the passport $[6^1 3^2 2^1 1^1, 2^4 1^7]$ and dessin d'enfant



(or its reflection for the other choice of a).

For $t = -405/4$ it has the passport $[4^3 2^1 1^1, 2^4 1^7]$, and dessin d'enfant



(or its reflection for the other choice of a).

We thank Michael Zieve, who communicated these special values of t to us in personal correspondence.

6.5 Polynomials Invertible in k-Radicals for $k \geq 15$

According to Theorem 9, polynomials invertible in k-radicals for $k \geq 15$ are compositions of power polynomials, Chebyshev polynomials and polynomials of degree at most k.

Thus there are no "exceptional" polynomials invertible in k-radicals for $k \geq 15$.

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